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Similarity reductions of partial differential equations

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Abstract. We determine the relation between two methods of reduction, the similarity reduction method using non-classical symmetry groups, and the direct approach used by Clarkson and Kruskal. We prove that the solutions which are obtained by similarity in correspondence to non-classical groups constitute a larger family than the one obtained by the method of Clarkson and Kruskal. The two procedures are equivalent only if the generators $\xi(x, t, u)$, $\tau(x, t, u)$, $\eta(x, t, u)$ of the non-classical groups are such that ξ/τ is independent of u. To explain these results, we prove the existence of families of solutions of the Burgers' equation which are found by means of non-classical symmetry reduction, and which cannot be determined via the general reduction form of Clarkson and Kruskal.

1. Introduction

The similarity reduction of a partial differential equation is the procedure by which it is possible to transform an equation having n variables to another in which only $n-1$ variables appear. It is obtained by means of a suitable choice of the variables and of the unknown function. Hence, for $n=2$, the similarity reduction transforms a partial differential equation into an ordinary differential equation and makes possible the determination of a class of solutions of the given partial differential equation depending on arbitrary constants.

A classical method to obtain similarity reductions is to use the symmetry properties of the equation: for any group of point symmetries admitted by the equation, it is possible to define a similarity reduction.

In [1], Clarkson and Kruskal (CK) propose a direct method to determine the similarity reduction for equations in two variables. It does not use techniques of group analysis and allows new classes of solutions for the Boussinesq equation to be found. With the same method, they also give the general reduction form for the Burgers' equation, the Kortweg–de Vries equation, and for the modified Kortweg–de Vries equation. Similarity reduction for a modified Boussinesq equation and for the BBM equation has been characterized by Clarkson in [2] and [3].

In a study to determine the connection between the direct method of CK and the group analysis technique using the Boussinesq equation, Levi and Winternitz [4] noticed that the solutions given by the CK direct reduction procedure were exactly those solutions obtained as invariant solutions under the non-classical symmetry groups [5] admitted by the equation.

They thus established the equivalence, for the Boussinesq equation, between the direct approach of CK and the reduction procedure using non-classical symmetry groups.

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Remarks on the relationship between the direct method and the non-classical symmetries are given also in [6].

In this note, we discuss the general problem of the correspondence between the two procedures.

The analysis given in [7] is essential. There, it has been noticed that the condition which characterizes the non-classical symmetries is the condition which establish the completeness of the system given by the differential equation and the invariance equation. This last equation is also interpretable as a 'side condition', according to the definition of Olver and Rosenau [8].

In [7] an algorithm is proposed which characterizes all the weak symmetries for a partial differential equation. The invariant solutions under weak symmetries which are not non-classical are obtained via a compatible system of ordinary equations in the same variable, and not by means of a single ordinary reduction equation. Therefore these weak symmetries cannot be related to the reduction procedure of CK. Nevertheless they are a valid tool for determination of exact solutions [9].

From the results of [7], we recognize that the solutions given by the direct method of CK are always given as invariant solutions under non-classical symmetries. There can, however, be solutions, obtained by similarity reduction with non-classical symmetries, which are not obtainable by means of the direct method of CK.

To explain these results, we consider the Burgers' equation and we determine non-classical groups and some corresponding similarity solutions.

Using some non-classical symmetries we determine two families of similarity solutions that are not obtainable by the general similarity reduction form of CK.

These solutions are also invariant for non-classical symmetries with time as the similarity variable. Therefore, we can also obtain these solutions via the special case of direct reduction (that is not included in the ansatz of CK), by means of which Lou [10] obtains new similarity solutions of the Boussinesq equation.

We point out that it is difficult to determine solutions for the Burgers' equation using the Lou approach. Indeed it is not possible to characterize, as for the Boussinesq equation, the special form of reduction that allows the determination of the general ordinary reduction equation.

2. Non-classical symmetry groups and completeness

Let us describe, briefly, the Bluman–Cole method for determining the non-classical symmetry groups. We will limit the discussion to the case of one nth order partial differential equation, on a function of two variables $u(x, t)$:

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{2.1}$$

It is known that a classical group of point symmetries which leaves (2.1) invariant may be determined by means of the vector fields $(\xi(x, t, u), \tau(x, t, u), \eta(x, t, u))$ of its generators. They are defined by the relation

$$\mathrm{pr}^{(n)}\Delta|_{\Delta=0} = 0. \tag{2.2}$$

Here $\mathrm{pr}^{(n)}$ indicates the nth prolongation of the transformations group and it is expressible by the derivatives of the generators [6].

Having defined the generators of a symmetry group, the invariance relation

$$I \equiv \xi(x, t, u)u_x + \tau(x, t, u)u_t - \eta(x, t, u) = 0 \tag{2.3}$$

allows determination of the similarity variable and the form of the similarity solution, using the general integral of the characteristic system.

The non-classical symmetry groups are defined as the transformation groups, which leave invariant the system Σ, given by (2.1), (2.3) and the differential consequences of (2.3), up to order n,

$$D_{hk}I = 0 \qquad h, k \in N_0, h+k = 1, \ldots, n-1. \tag{2.4}$$

D_{hk} is the operator corresponding to h total derivatives with respect to the variable t and k total derivatives with respect to the variable x.

Since this is identically

$$\mathrm{pr}^{(n)}I|_{I=0} = 0 \quad \text{and} \quad \mathrm{pr}^{(n)}D_{hk}I|_{I=0;D_{hk}I=0} = 0 \tag{2.5}$$

the condition which characterizes the non-classical group generators is

$$\mathrm{pr}^{(n)}\Delta|_\Sigma = 0. \tag{2.6}$$

This relation gives rise to the differential system (in general nonlinear) which defines the generators. With no loss of generality we can set $\tau = 1$ or $\tau = 0$.

Another way to obtain (2.6) is by examining the compatibility of the system of equations Σ; that is, by looking for the constraints on ξ, η, τ which force Σ to admit solutions.

These constraints are deduced by imposing the Schwarz conditions on the derivatives of order $n+1$.

As proved in [7], for the system Σ there is only one independent compatibility condition: once this condition (corresponding to the independence of one mixed derivative on the order of derivation) is satisfied, then the conditions of independence of all the other mixed derivatives are also verified.

By direct computation it is possible to see that the compatibility condition is satisfied, identically, as an algebraic consequence of Σ, if and only if (2.6) is verified.

Hence (2.6) is the condition which characterizes the completeness of the system Σ, because it assures that any differential consequence of Σ is also an algebraic consequence. When it holds, there exist solutions for the system Σ.

3. Clarkson and Kruskal's direct method and non-classical groups

The direct method proposed by CK consists in finding the similarity reduction of (2.1) in the form

$$u(x, t) = U(x, t, w(z(x, t))) \tag{3.1}$$

and, therefore, in looking for a form of U and z, such that, by replacement of (3.1) in (2.1), one obtains a differential equation in $w(z)$. Similarity solutions are obtained from (3.1) for U, z, and w values, which solve the equation (2.1).

In order to analyse the relation between the direct method and group analysis, it is helpful to make the following observation.

The vector fields $\xi(x, t, u)$, $\tau(x, t, u)$, $\eta(x, t, u)$, whose characteristic curves are the two-parameter family

$$z(x, t) = h \tag{3.2}$$

$$H(x, t, u) = w \tag{3.3}$$

(h and w arbitrary parameters) are exactly the ones for which ξ/τ is independent of u.

The relations (3.2), (3.3) are associated with vector fields defined by

$$\tilde{\xi}=\lambda\frac{\partial z}{\partial t}\frac{\partial H}{\partial u} \qquad \tilde{\tau}=\lambda\frac{\partial z}{\partial x}\frac{\partial H}{\partial u} \qquad \tilde{\eta}=\lambda\left(-\frac{\partial z}{\partial t}\frac{\partial H}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial H}{\partial t}\right) \tag{3.4}$$

and, hence, $\tilde{\xi}/\tilde{\tau}$ does not depend on u.

Vice versa, if ξ/τ is independent of u, the characteristic system

$$\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}u}{\eta} \tag{3.5}$$

has an integral of the form (3.2), which is obtained by integrating the equation $\mathrm{d}x/\mathrm{d}t=\xi/\tau$. (If $\partial z/\partial x=0$, that is, without lost of generality, if $z=t$, the previous observations hold for the ratio $\tilde{\tau}/\tilde{\xi}$.)

In the direct method (3.2) is assumed to be the similarity variable; when (3.1) is expressed in terms of w, it is a relation of the form (3.3).

Therefore, with the similarity variable and with the form (3.1) of similarity reduction, the direct method defines the quasilinear equation

$$\tilde{\xi}u_x+\tilde{\tau}u_t-\tilde{\eta}=0 \tag{3.6}$$

of which (3.1) is a general integral, and vice versa.

On these grounds, finding the similarity solutions by the direct method is equivalent to recognizing when (3.6) is *compatible with* (2.1), *with* $\tilde{\xi}$, $\tilde{\tau}$, $\tilde{\eta}$ *given by* (3.4).

From the preceding section, this is equivalent to finding the similarity solutions corresponding to non-classical symmetry groups of the form

$$\xi/\tau=f_1(t,x) \qquad \eta/\tau=f_2(x,t,u). \tag{3.7}$$

On the other hand, the special reduction form used by Lou [10]

$$u(x,t)=U(x,t,w(t))$$

is equivalent to determining the invariant solutions under non-classical groups with corresponding generators

$$\tau=0 \qquad \eta/\xi=f(x,t,u).$$

For a prescribed equation, the family of invariant solutions under non-classical groups is, in general, larger than that obtained with the CK method; in fact, it also contains the solutions corresponding to possible groups for which ξ/τ depends on u and the solutions corresponding to groups for which $\tau=0$.

For the Boussinesq equation all the non-classical symmetries with $\tau=1$ have ξ/τ independent of u. For this reason the family of invariant solutions determined in [4] coincides with the one obtained by CK.

4. Burgers' equation

To give an example, we consider the Burgers' equation

$$u_t+uu_x+u_{xx}=0. \tag{4.1}$$

By the direct method, Clarkson and Kruskal have obtained the general similarity reduction of the form [1]

$$z=\theta(t)x+\sigma(t) \qquad u(x,t)=\theta w(z)-\frac{1}{\theta}\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) \tag{4.2}$$

where $\theta(t)$ and $\sigma(t)$ are solutions of the equations

$$\theta\frac{d^2\sigma}{dt^2}-2\frac{d\theta}{dt}\frac{d\sigma}{dt}=\theta^5(A\sigma+2B) \qquad \left(\frac{d\theta}{dt}\right)^2=A\theta^6+C\theta^4 \tag{4.3}$$

and $\theta(t)\neq 0$. The corresponding equation for w is

$$\frac{d^2w}{dz^2}+w\frac{dw}{dz}-Az-2B=0. \tag{4.4}$$

In (4.3) and (4.4), A, B and C are arbitrary constants.

Equation (4.1) admits the classical symmetry groups defined by the generators

$$\xi_0=(at+b)x+ct+d \qquad \tau_0=at^2+2bt+e \qquad \eta_0=-(at+b)u+ax+c \tag{4.5}$$

and the non-classical symmetry groups, with generators defined by the system:

$$\tau=1 \qquad \xi_{uu}=0 \qquad \eta_{uu}-2\xi_{xu}+2(u-\xi)\xi_u=0 \tag{4.6}$$

$$\xi_{xx}+(2\xi-u)\xi_x-2\eta\xi_u+\xi_t-2\eta_{xu}-\eta=0 \tag{4.7}$$

$$2\eta\xi_x+u\eta_x+\eta_{xx}+\eta_t=0 \tag{4.8}$$

or by

$$\tau=0 \qquad \xi=1 \qquad u\eta_x+\eta_{xx}+\eta^2\eta_{uu}+2\eta\eta_{ux}+\eta_t+\eta^2=0. \tag{4.9}$$

The system (4.6), (4.7), (4.8) is proposed also in Ames [11], but solutions are determined here for the first time. From (4.6), one deduces that

$$\xi=\rho(x,t)u+\delta(x,t) \tag{4.10}$$

$$\eta=\tfrac{1}{3}\rho(\rho-1)u^3+(\rho_x+\rho\delta)u^2+\gamma_1(x,t)u+\gamma_2(x,t). \tag{4.11}$$

By a direct check of (4.7) and (4.8) the following three families of non-classical symmetry generators are defined:

(1)
$$\begin{aligned}&\xi=\alpha(t)x+\beta(t) \qquad \tau=1\\ &\eta=-\alpha u+x\left(\frac{d\alpha}{dt}+2\alpha^2\right)+2\alpha\beta+\frac{d\beta}{dt}\end{aligned} \tag{4.12}$$

with $\alpha(t)$ and $\beta(t)$ solutions of

$$a\exp\left(-2\int\alpha\,dt\right)=\frac{d\alpha}{dt}+2\alpha^2 \tag{4.13}$$

$$c\exp\left(-2\int\alpha\,dt\right)=2\alpha\beta+\frac{d\beta}{dt} \tag{4.14}$$

with a and c arbitrary parameters.

(2) $\quad \xi=u \qquad \tau=1 \qquad \eta=0.$

(3)
$$\begin{aligned}&\xi=-\tfrac{1}{2}u+g_0t^2+g_1t+g_2 \qquad \tau=1\\ &\eta=\tfrac{1}{4}u^3-\tfrac{1}{2}u^2(g_0t^2+g_1t+g_2)+(g_0t+\tfrac{1}{2}g_1)xu\\ &\qquad+(f_0t-\tfrac{1}{4}f^2)u-\tfrac{1}{2}g_0x^2-f_0x+g_0t+g_3\end{aligned}$$

with g_i, f, f_0 arbitrary parameters.

Additionally, there exists a fourth family defined by (4.9) for any η which solves the equation.

In the following we examine the solutions which correspond to the first three families.

(i) This family is the only one for which ξ/τ does not depend on u. If we let

$$\theta(t)=\exp\left(-2\int \alpha\,\mathrm{d}t\right) \qquad \sigma(t)=-\int \beta\theta\,\mathrm{d}t$$

that is

$$\alpha=-\frac{1}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad \beta=-\frac{1}{\theta}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

then (4.13) and (4.14) become, respectively

$$a\theta^4=3\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2-\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{4.15}$$

$$-c\theta^4=3\frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}-\theta\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2}. \tag{4.16}$$

These equations admit the integrals

$$at+\frac{1}{\theta^3}\frac{\mathrm{d}\theta}{\mathrm{d}t}=-b \quad \text{and} \quad \frac{1}{\theta^3}\frac{\mathrm{d}\sigma}{\mathrm{d}t}-ct=d$$

and, therefore

$$\theta^2(t)=(at^2+2bt+e)^{-1} \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}t}=\theta^3(d+ct).$$

Then, it follows that

$$\alpha=\frac{at+b}{at^2+2bt+e} \qquad \beta=\frac{ct+d}{at^2+2bt+e}. \tag{4.17}$$

Equations (4.15) and (4.16) are forms equivalent to (4.3): elimination of the parameters present in both equations results in the same system of equations of the third order on $\theta(t)$ and $\sigma(t)$. The form (4.15) and (4.16) is more useful, because it allows solution by integration.

The invariant similarity solutions under these non-classical symmetries are exactly the solutions obtained using the direct method of CK.

It is also evident, from (4.17), that this family of non-classical symmetries results in the same solutions obtained by means of classical symmetries; in fact $\xi=\xi_0/\tau_0$ and $\eta=\eta_0/\tau_0$.

(ii) The associated characteristic system is

$$\frac{\mathrm{d}x}{\mathrm{d}t}=u \qquad \frac{\mathrm{d}u}{\mathrm{d}t}=0.$$

The family of invariant solutions is defined, implicitly, by

$$x-ut-G(u)=0 \quad \text{with } G''=0$$

and therefore it is given by

$$u(x,t)=\frac{x+h_0}{t+h_1}$$

with h_0, h_1 arbitrary parameters.

Notice that these solutions are obtained also by the method of CK, when $A=B=0$, $w=0$, $\theta(t)=1/(t+h_1)$ and $\sigma(t)=h_0/(t+h_1)$, since they are invariants under the symmetries (4.12) with $\alpha(t)=1/(t+h_1)$ and $\beta(t)=h_0/(t+h_1)$.

(iii) Let us consider in detail the following three subcases:

$$(\text{iii}_a) \qquad \xi=-\tfrac{1}{2}u \qquad \tau=1 \qquad \eta=\frac{u^3}{4}$$

$$(\text{iii}_b) \qquad \xi=-\tfrac{1}{2}u \qquad \tau=1 \qquad \eta=\frac{u^3}{4}-f^2\frac{u}{4}$$

$$(\text{iii}_c) \qquad \xi=-\tfrac{1}{2}u+g \qquad \tau=1 \qquad \eta=\frac{u^3}{4}-g\frac{u^2}{2}.$$

(iii_a) The associated characteristic system is

$$\frac{-2\,\mathrm{d}x}{u}=\frac{\mathrm{d}t}{1}=\frac{4\,\mathrm{d}u}{u^3}.$$

Let us assume $z=t+2u^{-2}$ as the similarity variable, then the solutions are defined by

$$u(x+G(z))-2=0$$

with $G(z)$ a solution of $G''+G'^3=0$. Therefore, they are

$$u=\frac{4(x+a_1)}{x^2-2t+2xa_1-a_2}$$

with a_1 and a_2 arbitrary parameters.

Even this family of solutions can be obtained in a different way, using the direct method, when $A=1$, $B=C=0$, $\theta^2=(-2t-a_2-a_1^2)^{-1}$, $\sigma=a_1\theta$ and considering the following solution of (4.4)

$$w=\frac{(z^3+5z)}{(z^2+1)}.$$

(iii_b) The associated characteristic system is

$$\frac{-2\,\mathrm{d}x}{u}=\frac{\mathrm{d}t}{1}=\frac{4\,\mathrm{d}u}{u(u^2-f^2)}.$$

Let us assume $z=[\exp(-fx)(u+f)]/(u-f)$ as the similarity variable; the family of solutions is defined, implicity, by

$$\exp\left(\frac{f^2t}{2}\right)u^2-(u^2-f^2)G(z)=0 \tag{4.18}$$

with $G(z)$ a solution of

$$2z^2G\frac{\mathrm{d}^2G}{\mathrm{d}z^2}-z^2\left(\frac{\mathrm{d}G}{\mathrm{d}z}\right)^2+2zG\frac{\mathrm{d}G}{\mathrm{d}z}-G^2=0. \tag{4.19}$$

Using the substitution $(1/G)\,\mathrm{d}G/\mathrm{d}z=p$, one obtains the Riccati equation

$$\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{1}{2}p^2+\frac{1}{z}p-\frac{1}{2z^2}=0$$

The general integral of (4.19) is, therefore,

$$G(z)=\frac{b_1(z+b_2)^2}{z}.$$

Thus, we have the following three-parameter family of solutions to (4.1)

$$u^2\exp\left(\frac{f^2t}{2}\right)-b_1\exp(fx)[(u+f)\exp(-fx)+b_2(u-f)]^2=0 \tag{4.20}$$

and therefore

$$u(x,t)=\frac{fb_1^{1/2}[b_2\exp(fx)-1]}{b_1^{1/2}[b_2\exp(fx)+1]\pm\exp(\frac{1}{2}fx+\frac{1}{4}f^2t)}. \tag{4.21}$$

These solutions cannot be obtained by the general form of CK.

Differentiating (4.20) with respect to x and substituting for b_1, we obtain

$$u_x=-\frac{b_2\exp(fx)u(u-f)-u(u+f)}{2(b_2\exp(fx)-1)}. \tag{4.22}$$

This is the equation of which (4.20) is the general integral, considering t, f and b_2 as parameters. On the other hand, from (4.2):

$$u_x=\theta^2w'-\frac{\theta'}{\theta}. \tag{4.23}$$

Since (4.4) implies

$$w'=-\tfrac{1}{2}w^2+\tfrac{1}{2}Az^2+2Bz-D$$

after substitution for equation (4.2), we obtain

$$u_x=P_1u^2+P_2u+P_3 \tag{4.24}$$

where

$$\begin{gathered}P_1=-\tfrac{1}{2}\qquad P_2=-\frac{1}{\theta}\left(x\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)\\ P_3=-\frac{1}{2}\left(\frac{x}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{1}{\theta}\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2+\frac{A\theta^2}{2}(x\theta+\sigma)^2+B\theta^2(x\theta+\sigma)-D\theta^2-\frac{1}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}.\end{gathered} \tag{4.25}$$

Since (4.24) must coincide with (4.22), it follows that

$$P_2=\frac{f}{2}\frac{b_2\exp(fx)+1}{b_2\exp(fx)-1}\qquad\text{and}\qquad P_3=0. \tag{4.26}$$

Looking at the series expansion in x, it is clear that there exists no choice of $\theta(t)$ and $\sigma(t)$, such that the two forms (4.25) and (4.26) corresponding to P_2, are compatible for any x, f, b_2.

Functions (4.21) are invariant solutions also under the *non-classical* symmetry with $\tau=0$, $\xi=1$ and η defined by the RHS of (4.22); thus they are obtainable with the special Lou case of the direct method with

$$u(x,t)=U(x,t,w(t))=\frac{fb_1^{1/2}[b_2\exp(fx)-1]}{b_1^{1/2}[b_2\exp(fx)+1]\pm\exp(\frac{1}{2}fx)w}$$

where $w(t)=\exp(f^2t/4)$.

(iii_c) The associated characteristic system is

$$\frac{-2\,dx}{u-2g}=\frac{dt}{1}=\frac{4\,du}{u^2(u-2g)}.$$

Let us assume $z=-x+2/u$ as the similarity variable, the family of solutions is defined, implicitly, by

$$u\ln\left(\frac{u-2g}{u}\right)+2g-ug^2(t+H(z))=0 \tag{4.27}$$

with $H(z)$ a solution of

$$\frac{d^2H}{dz^2}+g^2\left(\frac{dH}{dz}\right)^2-2g\frac{dH}{dz}+1=0 \tag{4.28}$$

therefore

$$H(z)=\frac{1}{g^2}\ln(z+c_1)+\frac{z}{g}+c_2.$$

Thus, we have the following three-parameter family of solutions

$$u=\frac{2(1+g\exp(gx-g^2t-c_2g^2))}{x-c_1+\exp(gx-g^2t-c_2g^2)}.$$

As in (iii_b) these solutions cannot be obtained by the general reduction form (4.2). But they are obtainable by means of the special Lou case of the direct method with

$$u(x,t)=U(x,t,w(t))=\frac{2gw\exp(gx)+2}{w\exp(gx)+x-c_1}$$

where $w(t)=\exp(-g^2(t+c_2))$.

We remark that determination of the solutions in the form $u=U(x,t,w(t))$ is difficult. Substitution in (4.1) gives the relation

$$U_t+UU_x+U_{xx}+U_w\frac{dw}{dt}=0$$

which implies the reduction condition

$$U_t+UU_x+U_{xx}=\Gamma(w,t)U_w.$$

The U are then defined by the nonlinear equation

$$(U_t+UU_x+U_{xx})_xU_w-(U_t+UU_x+U_{xx})U_{wx}=0.$$

5. Conclusions

In this work we analyse the first of the questions posed by CK in the concluding discussion [1]. We establish the relation between the direct method which they propose and the generalization of Lie's classical method (Bluman and Cole [3]) using non-classical symmetry groups. In fact, we demonstrate that there is equivalence between the two methods only for those groups such that the ratio ξ/τ of the generators is independent of u and $\tau\neq 0$. The detailed analysis of similarity solutions associated with non-classical groups of Burgers' equation indicate the following result: the reduction method with non-classical groups has more general results, since there exist two classes of solutions that cannot be obtained by the direct method of CK. These solutions have the form of the special reduction proposed by Lou.

In general we cannot claim that invariant solutions under non-classical groups with ξ/τ dependent on u are also invariant under groups with ξ/τ independent of u or with $\tau=0$. We can, however, conclude that the invariant solutions found under non-classical groups include those found via the direct methods.

References

[1] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201-13
[2] Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 2355-67
[3] Clarkson P A 1989 *J. Phys. A: Math. Gen.* **22** 3821-48
[4] Levi D and Winternitz P 1989 *J. Phys. A: Math. Gen.* **22** 2915-24
[5] Bluman G W and Cole J D 1969 *J. Math. Mech.* **18** 1025-42
[6] Clarkson P A and Winternitz P 1991 *Physica* **49D** 257-72
[7] Pucci E and Saccomandi G 1992 *J. Math. Anal. Appl.* **163** 588-98
[8] Olver P and Rosenau P 1986 *Phys. Lett.* **114A** 107-12
[9] Olver P J 1986 *Application of Lie Groups of Differential Equations* (Berlin: Springer)
[10] Lou S 1990 *Phys. Lett.* **151A** 133-5
[11] Ames W F 1972 *Nonlinear Partial Differential Equations in Engineering* (New York: Academic)